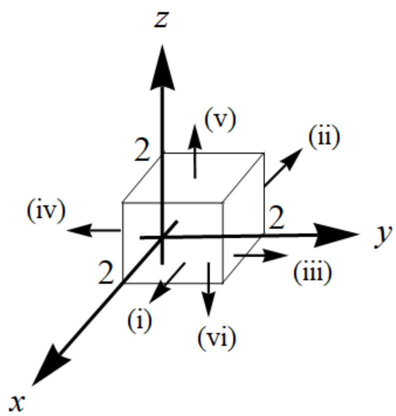
Problem 1.30

Calculate the surface integral of the function in Ex. 1.7, over the *bottom* of the box. For consistency, let "upward" be the positive direction. Does the surface integral depend only on the boundary line for this function? What is the total flux over the *closed* surface of the box (*including* the bottom)? [*Note:* For the *closed* surface, the positive direction is "outward," and hence "down," for the bottom face.]

Solution

The box of interest here is shown below.



Let

$$\mathbf{v} = 2xz\hat{\mathbf{x}} + (x+2)\hat{\mathbf{y}} + y(z^2-3)\hat{\mathbf{z}}$$

and calculate the integral of $\mathbf{v}$ over each of the faces.

$$\iint_{\text{face (i)}} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 \mathbf{v} \cdot (\hat{\mathbf{x}}\, dy\, dz)\bigg|_{x=2} = \int_0^2 \int_0^2 v_x\bigg|_{x=2} dy\, dz = \int_0^2 \int_0^2 2(2)z\, dy\, dz = 16$$

$$\iint_{\text{face (ii)}} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 \mathbf{v} \cdot (-\hat{\mathbf{x}}\, dy\, dz)\bigg|_{x=0} = -\int_0^2 \int_0^2 v_x\bigg|_{x=0} dy\, dz = -\int_0^2 \int_0^2 2(0)z\, dy\, dz = 0$$

$$\iint_{\text{face (iii)}} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 \mathbf{v} \cdot (\hat{\mathbf{y}}\, dx\, dz)\bigg|_{y=2} = \int_0^2 \int_0^2 v_y\bigg|_{y=2} dx\, dz = \int_0^2 \int_0^2 (x+2)\, dx\, dz = 12$$

$$\iint_{\text{face (iv)}} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 \mathbf{v} \cdot (-\hat{\mathbf{y}}\, dx\, dz)\bigg|_{y=0} = -\int_0^2 \int_0^2 v_y\bigg|_{y=0} dx\, dz = -\int_0^2 \int_0^2 (x+2)\, dx\, dz = -12$$

$$\iint_{\text{face (v)}} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 \mathbf{v} \cdot (\hat{\mathbf{z}}\, dx\, dy)\bigg|_{z=2} = \int_0^2 \int_0^2 v_z\bigg|_{z=2} dx\, dy = \int_0^2 \int_0^2 y(1)\, dx\, dy = 4$$

$$\iint_{\text{face (vi)}} \mathbf{v} \cdot d\mathbf{a} = \int_0^2 \int_0^2 \mathbf{v} \cdot (-\hat{\mathbf{z}}\, dx\, dy)\bigg|_{z=0} = -\int_0^2 \int_0^2 v_z\bigg|_{z=0} dx\, dy = -\int_0^2 \int_0^2 y(-3)\, dx\, dy = 12$$

Therefore, the closed surface integral of $\mathbf{v}$ over the box is

$$\begin{aligned}\oiint\limits_{\text{box}} \mathbf{v}\cdot d\mathbf{a} &= \iint\limits_{\text{face (i)}} \mathbf{v}\cdot d\mathbf{a} + \iint\limits_{\text{face (ii)}} \mathbf{v}\cdot d\mathbf{a} + \iint\limits_{\text{face (iii)}} \mathbf{v}\cdot d\mathbf{a} + \iint\limits_{\text{face (iv)}} \mathbf{v}\cdot d\mathbf{a} + \iint\limits_{\text{face (v)}} \mathbf{v}\cdot d\mathbf{a} + \iint\limits_{\text{face (vi)}} \mathbf{v}\cdot d\mathbf{a} \\ &= 16 + 0 + 12 - 12 + 4 + 12 \\ &= 32.\end{aligned}$$

Because it's nonzero, the integral of $\mathbf{v}$ over the bottom face does not only depend on the boundary line; the surface being integrated over also matters. This conclusion could have been reached more quickly from the fact that the divergence of $\mathbf{v}$ is nonzero.

$$\begin{aligned}\nabla\cdot\mathbf{v} &= \frac{\partial}{\partial x}(2xz) + \frac{\partial}{\partial y}(x+2) + \frac{\partial}{\partial z}[y(z^2-3)] \\ &= 2z + 0 + 2yz \\ &= 2z(1+y)\end{aligned}$$

Observe that the volume integral of $\nabla\cdot\mathbf{v}$ over the box gives the same result, verifying Gauss's theorem.

$$\begin{aligned}\iiint\limits_{\text{box}} \nabla\cdot\mathbf{v}\, dV &= \int_0^2\int_0^2\int_0^2 2z(1+y)\, dx\, dy\, dz \\ &= 2\left(\int_0^2 dx\right)\left[\int_0^2 (1+y)\, dy\right]\left(\int_0^2 z\, dz\right) \\ &= 2(2)(4)(2) \\ &= 32\end{aligned}$$